

Mathematics Framework Solution Sets

Geometry

A note about these solutions.

These solutions are intended for teachers, not students. The solutions are fairly detailed and include additional comments that serve to further explain the content and purpose of each problem. It is important to note that the solutions are not meant to be representative of student solutions. Also, do not assume that any diagrams are drawn to scale.

It is the nature of many mathematics problems that they can be solved in different ways. The solutions given here represent simply one way of solving the problems. In particular, problems in Geometry often lend themselves to more than one solution, since there is an intrinsic dependence on what you are assuming as axioms and which theorems have been proven and are therefore usable.

It is our hope that these solution sets help teachers to better see the essential skills and concepts that are important to student success in Geometry.

A note about notation.

Throughout these solutions, points are represented by capital letters. Also, AB is used to denote the segment connecting points A and B , while \overline{AB} is used to denote the length of segment AB . Therefore, we will write $AB \cong CD$ to mean that segment AB is congruent to CD , and $\overline{AB} = \overline{CD}$ to mean that they have the same length. (A necessary distinction.)

A similar convention holds for $\angle ABC$ and $m\angle ABC$, wherein the former denotes the angle itself and the latter denotes the measure of the angle.

Also, \overleftrightarrow{AB} will be used to denote the unique line through the points A and B , of which the segment AB is a part. By a slight abuse of notation, we will write $AB \parallel CD$ to mean that segments AB and CD are parallel (it is more appropriate to state AB and CD are parts of the parallel lines, \overleftrightarrow{AB} and \overleftrightarrow{CD}).

3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Problem: Prove or disprove: Any two right triangles with the same hypotenuse have the same area.

Solution: We can disprove this statement by finding two right triangles that have hypotenuses of the same length but have different areas. Let $\triangle ABC$ be a right triangle with legs of length 3 and 4. Then by the Pythagorean theorem, the hypotenuse of $\triangle ABC$ must have length 5 since $3^2 + 4^2 = 5^2$. Similarly, let $\triangle DEF$ be a right triangle with one leg of length 2. Then by the Pythagorean theorem, if the hypotenuse of $\triangle DEF$ is to have length 5, the other leg must measure

$$\sqrt{5^2 - 2^2} = \sqrt{25 - 4} = \sqrt{21}.$$

Since the legs of a right triangle are perpendicular, either may serve as the base of the triangle if the other serves as the height. Thus the area of triangle $\triangle ABC$ is

$$\text{Area}(\triangle ABC) = \frac{1}{2} \cdot 3 \cdot 4 = 6,$$

whereas the area of triangle $\triangle DEF$ is

$$\text{Area}(\triangle DEF) = \frac{1}{2} \cdot 2 \cdot \sqrt{21} = \sqrt{21}.$$

Clearly, $6 \neq \sqrt{21}$, and so the statement is false. That is, any two right triangles with the same hypotenuse do not necessarily have the same area.

Further Explanation: A common fallacy in proving a statement is to attempt to exhibit an example of the statement to serve as a proof. Typically this is insufficient to prove a statement correct as one example cannot prove a general statement true for all cases. However, to *disprove* a statement, it is sufficient to simply provide one case in which the statement is false, as in the problem above. Such an example is known as a *counterexample*.

3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Problem: True or false? A quadrilateral is a rectangle only if it is a square.

Solution: By definition, a rectangle is a quadrilateral with each interior angle a right angle, having measure 90° . A square by definition is a quadrilateral with each interior angle a right angle that has congruent sides. Since a quadrilateral can have interior angles all measuring 90° without having sides of the same length, clearly this statement is false.

Further Explanation: This problem is an exercise in understanding definitions and the classification of geometric objects. Common mistakes in classification include stating that squares are not rectangles because they have an added property (they are), or that rectangles are squares (they are not; they lack congruent sides).

3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Problem: Suppose that all triangles that satisfy property A are right triangles. Is the following true or false? A triangle that does not satisfy the Pythagorean theorem does not satisfy property A .

Solution: Suppose triangle $\triangle ABC$ does not satisfy the Pythagorean theorem. Then $\triangle ABC$ cannot be a right triangle. If it is not a right triangle, then it cannot satisfy property A . Therefore this statement is true.

Further Explanation: There are two main examples of logical reasoning present in this problem. One is in understanding the equivalence of the contrapositive of an “if-then” statement. The *contrapositive* of a statement “If P , then Q ,” is the statement “If *not* Q , then *not* P .” The contrapositive is always logically equivalent to the original statement. In the problem above, the contrapositive of the statement,

“If a triangle satisfies property A , then it is a right triangle,”

is logically equivalent to the statement

“If a triangle is *not* a right triangle, then it does *not* satisfy property A .”

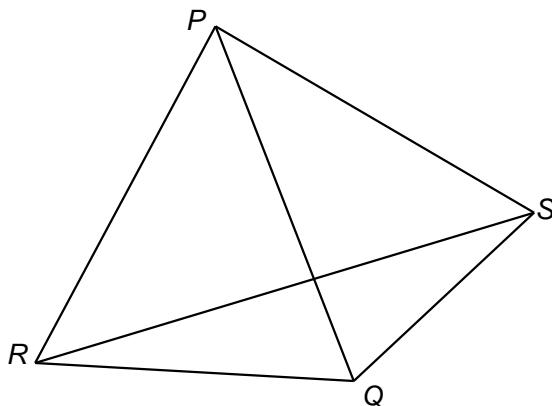
Second, one must understand the “if-and-only-if” nature of the Pythagorean theorem. It states:

“A triangle is a right triangle if and only if the sum of the squares of the lengths of the legs is equal to the square of the hypotenuse.”

Therefore, if the Pythagorean formula $a^2 + b^2 = c^2$ is not satisfied, then the triangle is not a right triangle.

4.0 Students prove basic theorems involving congruence and similarity.

Problem: Suppose that triangle $\triangle PRS$ is isosceles, with $\overline{RP} = \overline{PS}$. Show that if the segment PQ bisects angle $\angle RPS$, then $\overline{RQ} = \overline{QS}$.



Solution: Since segment PQ bisects angle $\angle RPS$, we have

$$\angle RPQ \cong \angle SPQ.$$

We know that $RP \cong SP$, since they have the same measure by hypothesis. Of course, $PQ \cong PQ$. Therefore

$$\triangle RPQ \cong \triangle SPQ,$$

by the Side-Angle-Side (SAS) criterion for triangle congruence. Finally, since corresponding parts of congruent triangles are congruent, we must have $RQ \cong QS$. Therefore $\overline{RQ} = \overline{QS}$.

Further Explanation: The Side-Angle-Side criterion for triangle congruence states that if two sides and the included angle of two triangles are congruent, then the triangles themselves are congruent.

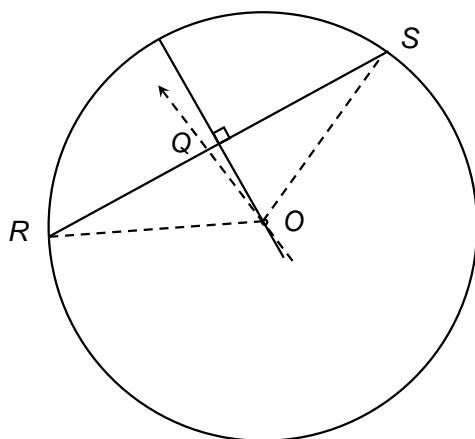
4.0 Students prove basic theorems involving congruence and similarity.

Problem: Suppose that R and S are points on a circle. Prove that the perpendicular bisector of the line segment RS passes through the center of the circle.

Solution: Consider the angle $\angle ROS$ where O is the center of the circle on which R and S lie. Construct the angle bisector of $\angle ROS$, and let this meet the segment RS at point Q . We will show that $\angle RQO$ is a right angle and that $RQ \cong QS$, which will show that O lies on the perpendicular bisector of RS .

Consider the triangles $\triangle ROQ$ and $\triangle SOQ$. We have that $\angle ROQ \cong \angle SOQ$ by construction of the angle bisector. Moreover, $OR \cong OS$ since R and S are on the circle and are therefore equidistant from O . Finally, $OQ \cong OQ$. Thus by SAS,

$$\triangle ROQ \cong \triangle SOQ.$$



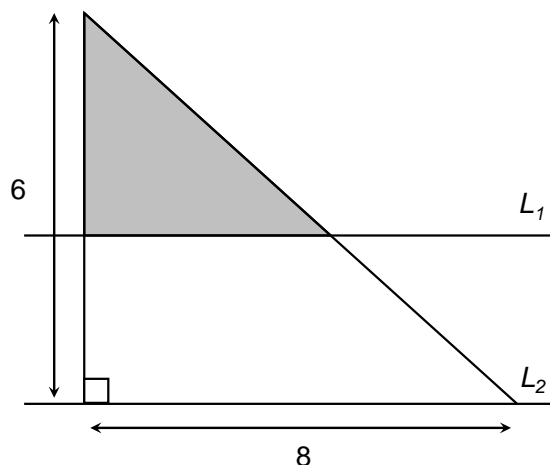
Finally, since corresponding parts of congruent triangles are congruent, we see that $RQ \cong QS$. Also, $\angle RQO \cong \angle SQO$. But since they are also supplementary, we have that

$$180^\circ = m\angle RQO + m\angle SQO = 2 \times m\angle RQO.$$

This implies that $m\angle RQO = 90^\circ$, or that $\angle RQO$ is a right angle. Since perpendicular bisectors are unique, this is sufficient to show that segment OQ is the perpendicular bisector of RS .

5.0 Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles.

Problem: In the figure shown below, the area of the shaded right triangle is 6. Find the distance between the parallel lines L_1 and L_2 . Explain your reasoning.



Solution: The shaded triangle and the larger triangle share the top angle in the picture, and since both are right triangles, their remaining angles must be congruent since the sum of the measures of the angles of a triangle is 180° . Therefore, the triangles are similar.

Let h be the height of the shaded triangle and b the length of its base. Since the length of the base of the larger triangle is $\frac{4}{3}$ times its height, we must also have $b = \frac{4}{3}h$. Since the area of the shaded triangle is 6, we have

$$6 = \frac{1}{2}b \cdot h = \frac{1}{2} \cdot \frac{4}{3}h \cdot h.$$

This simplifies to

$$\frac{2}{3}h^2 = 6 \Rightarrow h^2 = 9.$$

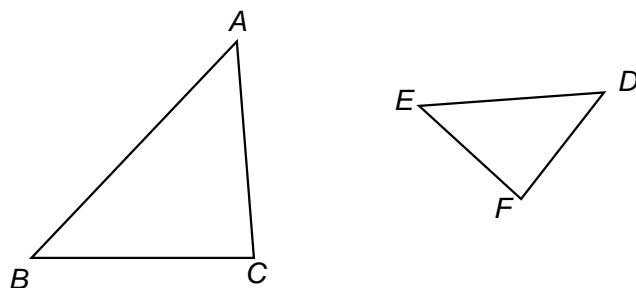
Thus the height of the shaded triangle is 3. Therefore, the distance between the parallel lines is $6 - 3 = 3$.

Further Explanation: If all of the corresponding pairs of angles of two triangles are congruent, then the triangles are similar. Once two triangles are similar, their side lengths share many ratios in common, including between the two triangles and within the two triangles. For example, if $\triangle ABC \sim \triangle DEF$, then

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}},$$

and therefore

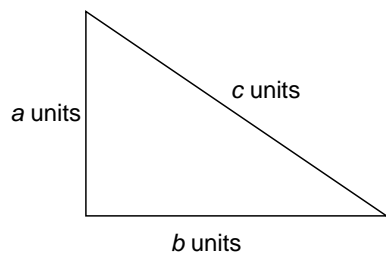
$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{DE}}{\overline{DF}}, \quad \frac{\overline{AB}}{\overline{BC}} = \frac{\overline{DE}}{\overline{EF}}, \quad \frac{\overline{BC}}{\overline{AC}} = \frac{\overline{EF}}{\overline{DF}}.$$



6.0 Students know and are able to use the triangle inequality theorem.

Problem: Using a geometric diagram, show that for any positive numbers a and b , $\sqrt{a^2 + b^2} < a + b$.

Solution: Since a and b are positive numbers, we may construct a right triangle with legs of length a and b units.



By the Pythagorean theorem, we know that

$$a^2 + b^2 = c^2 \Rightarrow c = \sqrt{a^2 + b^2},$$

since c must be positive as well. The triangle inequality theorem states that the length of any given side of a triangle is less than the sums of the lengths of the other two remaining sides. Thus, we have

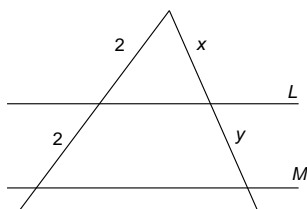
$$c < a + b,$$

or by substitution,

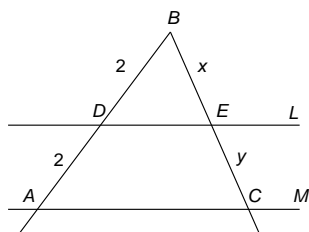
$$\sqrt{a^2 + b^2} < a + b.$$

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Problem: On the following diagram, with distances as shown, prove that if $x = y$ then the lines L and M are parallel.



Solution: Label the intersections of the various lines in the diagram $A - E$ as shown.



We are given that $x = y$, and so we see that

$$\frac{2}{2} = \frac{\overline{DB}}{\overline{AD}} = \frac{\overline{EB}}{\overline{EC}} = \frac{x}{y}.$$

Adding 1 to each side of the equation, we have

$$\frac{\overline{AB}}{\overline{AD}} + \frac{\overline{DB}}{\overline{AD}} = \frac{\overline{EC}}{\overline{EC}} + \frac{\overline{EB}}{\overline{EC}},$$

or,

$$\frac{\overline{AD} + \overline{DB}}{\overline{AD}} = \frac{\overline{EC} + \overline{EB}}{\overline{EC}}.$$

Since $AD + DB = AB$ and $EC + EB = BC$ by segment addition, we can write

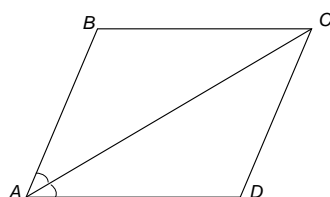
$$\frac{\overline{AB}}{\overline{AD}} = \frac{\overline{BC}}{\overline{EC}}.$$

But this means that line L divides the sides AB and BC proportionally. By a well-known theorem, L must be parallel to the third side AC of $\triangle ABC$. Since AC is part of line M , we must have $L \parallel M$.

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Problem: Prove that if a diagonal of a parallelogram bisects an angle of the parallelogram, then the parallelogram is a rhombus.

Solution: Label the parallelogram $ABCD$ so that A is opposite C and construct diagonal AC as shown.



Assuming that AC bisects $\angle BAD$, we have

$$\angle DAC \cong \angle BAC.$$

Since opposite angles are congruent in a parallelogram, we have

$$\angle ABC \cong \angle ADC.$$

Since the angle sum of any triangle is 180° , we have

$$m\angle BAC + m\angle ABC + m\angle BCA = 180^\circ = m\angle DAC + m\angle ADC + m\angle DCA.$$

By the congruences above, we can subtract angles of equal measure from each side to conclude that $m\angle BCA = m\angle DCA$ also. Clearly, $AC \cong AC$, so that by Angle-Side-Angle (ASA) we have

$$\triangle ABC \cong \triangle ADC.$$

Therefore, since corresponding parts of congruent triangles are congruent, we can conclude that $AB \cong AD$. But $AB \cong CD$ and $AD \cong CB$ since $ABCD$ is a parallelogram. Applying the transitivity of congruence, we have

$$AB \cong AD \cong CD \cong CB,$$

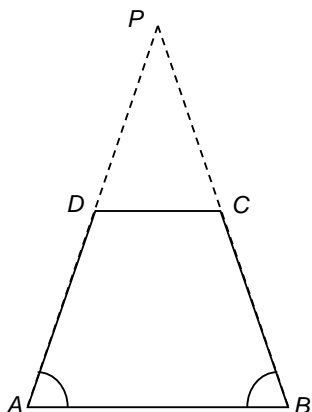
and therefore $ABCD$ is a rhombus.

Further Explanation: The Angle-Side-Angle (ASA) criterion for triangle congruence states that if two angles and the include side of two triangles are congruent, then so are the triangles.

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Problem: Prove that if the base angles of a trapezoid are congruent, then the trapezoid is isosceles.

Solution: Suppose that $ABCD$ is a trapezoid labeled as shown.



Extend segments AD and BC to meet at a point P beyond DC , which is possible since by definition of a trapezoid, AD and BC are not parallel and therefore \overrightarrow{AD} and \overrightarrow{BC} intersect at a unique point. By assumption $\angle A \cong \angle B$, so that $\triangle APB$ is isosceles. Thus $AP \cong BP$ since the sides adjacent to base angles in an isosceles triangle are congruent. Since $ABCD$ is a trapezoid, we have that $AB \parallel CD$. Therefore, since when parallel lines are cut by a transversal the resulting corresponding angles are congruent, we can say

$$\angle A \cong \angle PDC \quad \text{and} \quad \angle B \cong \angle PCD.$$

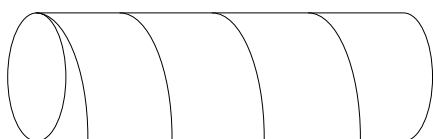
Thus $\angle PDC \cong \angle PCD$. This means that $\triangle DPC$ is isosceles as well, so that $PD \cong PC$. Finally, if we subtract congruent segments from congruent segments, we obtain

$$AP - DP \cong BP - CP \quad \Rightarrow \quad AD \cong BC.$$

This shows that trapezoid $ABCD$ is isosceles.

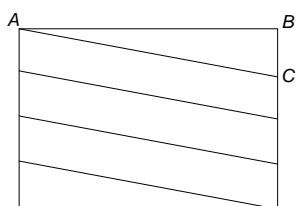
8.0 Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

Problem: A string is wound, evenly-spaced, around a circular rod. The string goes exactly one time around the rod. The circumference of the rod is 4 cm, and its length is 12 cm. Find the length of the string. What is the length of the string if it goes exactly four times around the rod? (Adapted from TIMMS fr, 12m K-14).



Solution: Cut the rod along a line through the starting point of the string and the ending point of the string to form the rectangle shown below.

We've used the assumption that the string is evenly



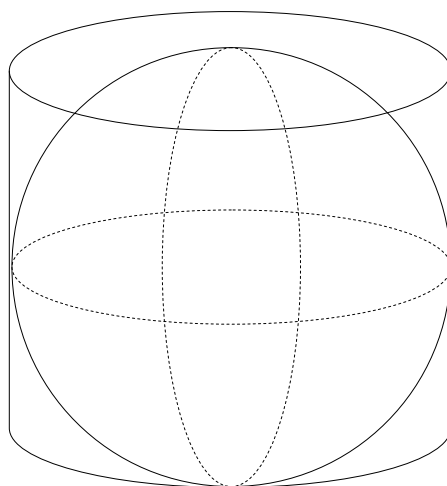
spaced and that it goes around the cylinder exactly four times. Consider the triangle $\triangle ABC$ indicated in the picture. Since the rod is 4 cm long, $\overline{BC} = 1$ cm. Also, since the circumference of the rod is 12 cm, we have $\overline{AB} = 12$ cm. Therefore

$$\overline{AC} = \sqrt{1^2 + 12^2} = \sqrt{145} \text{ cm.}$$

Since this length occurs 4 times around the rod, the total length of the string must be $4\sqrt{145} \approx 48.166$ cm.

9.0 Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

Problem: A sphere of radius 1 can be inscribed in a cylinder so that it touches the top face, bottom face, and intersects the lateral face in a circle. Find the volume of the cylinder.



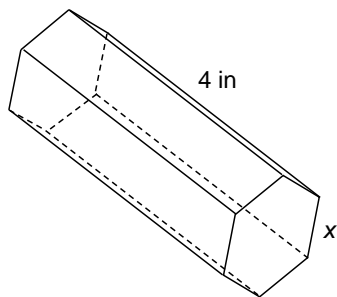
Solution: The intersection of the sphere and the lateral face (the label of the soup can if you like) is a circle of radius 1. Therefore, the radii of the base circles (the top or bottom of the can) are also 1. The height of the cylinder is the diameter of the sphere, which is 2. Therefore, the volume can be found by computing

$$V = \pi r^2 h = \pi \cdot 1^2 \cdot 2 = 2\pi.$$

9.0 Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

Problem: A right prism with a 4-inch height has a regular hexagonal base. The prism has a volume of 144 cubic inches. Find the surface area of the prism.

Solution: Such a prism is shown below.

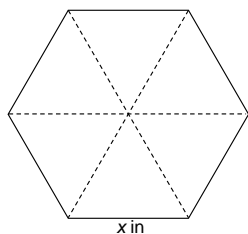


The height of the prism is given as 4 in. The surface area of the prism is the sum of the areas of the six rectangular faces and the areas of the two hexagonal bases. Label the length of one of the sides of the base as x in and denote the area of the hexagonal base of the prism by B . Then the surface area S can be found by

$$S = 2B + 6(4x) = 2B + 24x.$$

The volume of the prism is given by $V = B \cdot h$. Therefore $144 \text{ in}^3 = 4B$, so that $B = 36 \text{ in}^2$.

Now, the regular hexagonal base can be divided into six congruent equilateral triangles as shown in the diagram below.



(Continued on next page.)

The area of one such triangle is 6 in^2 since the area of the base is 36 in^2 . A general formula for the area of an equilateral triangle is

$$A = \frac{x^2\sqrt{3}}{4}.$$

Thus, we can substitute to find x :

$$6 = \frac{x^2\sqrt{3}}{4} \Rightarrow x^2 = \frac{24}{\sqrt{3}}.$$

Simplifying, we obtain

$$x^2 = \frac{24}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3},$$

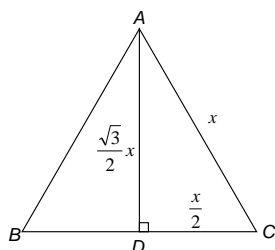
or $x = \sqrt{8\sqrt{3}} = 2\sqrt{2\sqrt{3}}$ in.

Finally, the surface area of the prism is then

$$S = 2 \cdot 36 + 24 \cdot 2\sqrt{2\sqrt{3}} \approx 161.34.$$

Thus the surface area is approximately 161.34 in^2 .

Further Explanation: Suppose we have an equilateral triangle $\triangle ABC$, with sides of length x units. Drop the perpendicular from angle A to BC . Then this bisects side BC into segments BD and DC .



By the Pythagorean theorem, $(\overline{AC})^2 = (\overline{AD})^2 + (\overline{CD})^2$ so that $x^2 = (\overline{AD})^2 + \frac{x^2}{4}$. Thus $(\overline{AD})^2 = \frac{3x^2}{4}$, which implies $\overline{AD} = \frac{x\sqrt{3}}{2}$. This shows that the height of the triangle is $x\sqrt{3}/2$, and since the base is x , the area can be found by

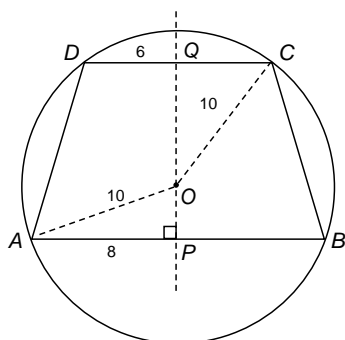
$$A = \frac{1}{2} \cdot x \cdot \frac{x\sqrt{3}}{2} = \frac{x^2\sqrt{3}}{4}.$$

Incidentally, since the interior angles of an equilateral triangle measure 60° , we see that $\triangle ADC$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. The diagram above explains why the ratios of the side lengths are $1 : \sqrt{3} : 2$.

10.0 Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids.

Problem: A trapezoid with bases of length 12 and 16 is inscribed in a circle of radius 10. The center of the circle lies inside the trapezoid. Find the area of the trapezoid.

Solution: Suppose $ABCD$ is a trapezoid with bases of lengths $\overline{AB} = 16$ and $\overline{CD} = 12$. Construct radii from the center of the circle O to vertices A and C as shown.



Construct the perpendicular bisector of AB ; it passes through the center of the circle since all points equidistant from A and B must lie on it. Label the points where it intersects AB and CD as P and Q respectively. Note that PQ is also perpendicular to CD since $AB \parallel CD$. Since by a general result, the perpendicular bisector of CD must pass through the center of the circle, this must be PQ . Therefore Q is the midpoint of CD .

By construction $\overline{AP} = 8$ and $\overline{CQ} = 6$. Also, triangles $\triangle OQC$ and $\triangle AOP$ are both right triangles by construction, so that

$$(\overline{AP})^2 + (\overline{OP})^2 = (\overline{AO})^2 \quad (1)$$

$$(\overline{CQ})^2 + (\overline{OQ})^2 = (\overline{CO})^2 \quad (2).$$

By equation (1), we have $(\overline{OP})^2 = 100 - 64 = 36$, so that $\overline{OP} = 6$. Similarly, by equation (2) $\overline{OQ} = 8$. Therefore, the height of the trapezoid is given by

$$\overline{PQ} = \overline{OP} + \overline{OQ} = 14.$$

Finally, the area is given by the familiar formula

$$A = \left(\frac{b_1 + b_2}{2} \right) h = \left(\frac{16 + 12}{2} \right) \cdot 14 = 196.$$

11.0 Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.

Problem: Brighto soap powder is packed in cube-shaped cartons. A carton measures 10 cm on each side. The company decides to increase the length of each edge of the carton by 10 percent. How much does the volume increase?

Solution: Let V_1 denote the original volume and V_2 denote the volume after increasing the lengths of each edge by 10%. The original cube has volume $V_1 = (10 \text{ cm})^3 = 1,000 \text{ cm}^3$. If we increase each side length by 10% this is equivalent to increasing them to

$$10 \text{ cm} + (0.10)(10 \text{ cm}) = 11 \text{ cm}.$$

Therefore, $V_2 = (11 \text{ cm})^3 = 1,331 \text{ cm}^3$. The original volume increased by 331 cm^3 . Equivalently, the percent increase of the volume was

$$\frac{V_2 - V_1}{V_1} = \frac{1,331 - 1,000}{1,000} = \frac{331}{1,000} = 0.331,$$

or 33.1%.

12.0 Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

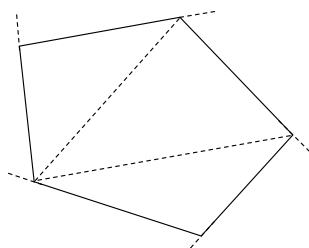
Problem: A regular polygon has exterior angles, each measuring 10 degrees. How many sides does the polygon have?

Solution: The sum of the measures of the exterior angles of any polygon is 360° . If the polygon has n sides, then it also has n vertices and angles, and the interior angles have the same measure by hypothesis. Therefore,

$$n \cdot 10^\circ = 360^\circ \Rightarrow n = 36.$$

Thus, the polygon has 36 sides.

Further Explanation: By a familiar construction, one can construct diagonals of an n -gon in such a way as to create $n - 2$ triangles interior to the polygon. The pentagon below is an example.



This picture illustrates the fact that the sum of the measures of the interior angles of an n -gon is $(n - 2) \cdot 180^\circ$. Notice that each exterior angle is the supplement of an interior angle of the polygon. Since there are n of them, we can find their sum by finding

$$n \cdot 180^\circ - (n - 2) \cdot 180^\circ = 360^\circ.$$

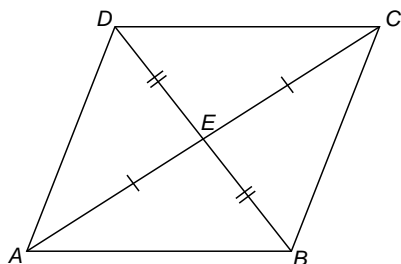
Thus, the sum of the exterior angles of a polygon is 360° .

The sum of the exterior angles of a polygon can be considered the amount of angle one turns through as they trace the polygon completely. After the tracing is finished, since one returns back to where they started, they must have traveled through 360° .

13.0 Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical, and exterior angles.

Problem: Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution: Label the quadrilateral as shown and let point E be the intersection of the diagonals AC and BD .



By assumption, $AE \cong EC$ and $BE \cong ED$. Also, $\angle AEB \cong \angle CED$ and $\angle AED \cong \angle CEB$, since vertical angles are congruent. Therefore,

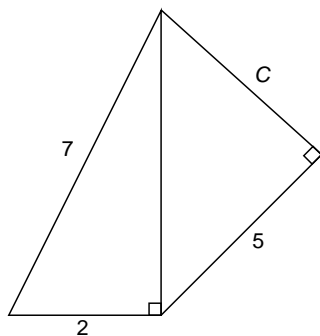
$$\triangle AEB \cong \triangle CED \quad \text{and} \quad \triangle AED \cong \triangle CEB$$

by SAS. Since corresponding angles of congruent triangles are congruent, we have $\angle CDE \cong \angle ABE$. But then the alternate interior angles of the lines \overleftrightarrow{AB} and \overleftrightarrow{CD} created by the transversal DB are congruent. This implies that $AB \parallel CD$. Similarly, $AD \parallel CB$ and therefore $ABCD$ is a parallelogram.

Further Explanation: It is important to note that in Euclidean Geometry, the Alternate Interior Angle theorem is an if-and-only-if statement. That is, two lines l and m are parallel if and only if the alternate interior angles formed by a transversal t are congruent to each other.

15.0 Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

Problem: Find the length of the side labeled C in the figure shown below:



Solution: If the remaining unlabeled side is A , then by the Pythagorean theorem,

$$2^2 + A^2 = 7^2 \Rightarrow A^2 = 49 - 4 = 45.$$

Also by the Pythagorean theorem,

$$C^2 + 5^2 = A^2,$$

so that

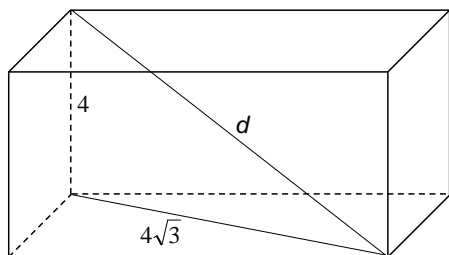
$$C^2 = A^2 - 25 = 45 - 25 = 20.$$

Therefore $C = \sqrt{20} = 2\sqrt{5}$.

15.0 Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

Problem: The bottom of a rectangular box is a rectangle with a diagonal whose length is $4\sqrt{3}$ inches. The height of the box is 4 inches. Find the length of a diagonal of the box.

Solution: The box is shown below.



The height, diagonal of the bottom of the box, and diagonal of the entire box form a right triangle. Therefore,

$$d^2 = 4^2 + (4\sqrt{3})^2 = 16 + 16 \cdot 3 = 16 + 48 = 64,$$

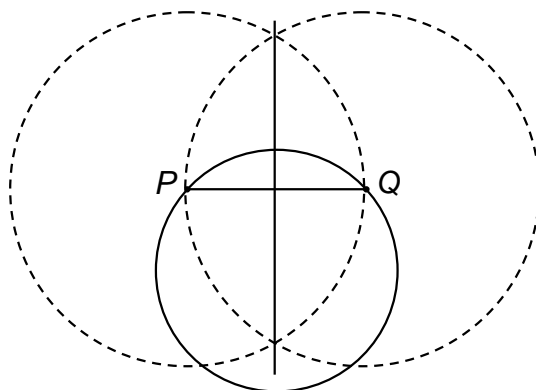
so that $d = 8$ in.

16.0 Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

Problem: Given a circle, use an unmarked straightedge and a compass to find the center.

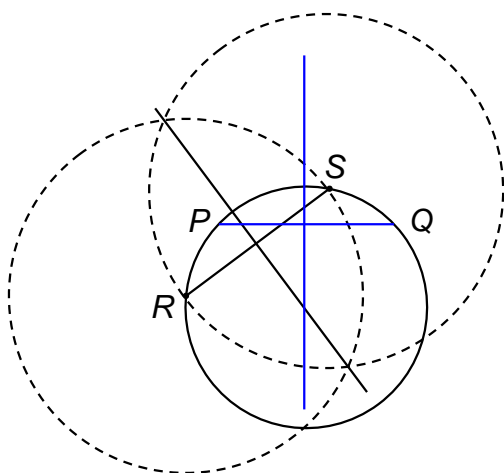
Solution: First, choose two points on the circle, P and Q . We construct the perpendicular bisector of segment PQ as follows:

Set the compass to the length \overline{PQ} and construct the circle of radius \overline{PQ} centered at P and then the circle of radius \overline{PQ} centered at Q . These two circles intersect in two distinct points. The line passing through these two points is the perpendicular bisector of PQ .

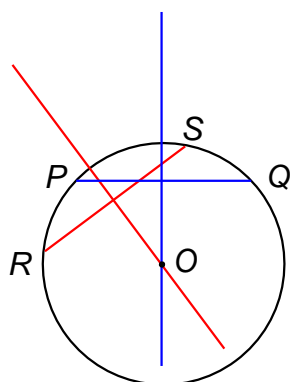


The perpendicular bisector of PQ contains all points equidistant from P and Q and so it contains the center of the circle. (*Continued on next page.*)

Now choose two new points R and S on the circle, and repeat the construction above to obtain the perpendicular bisector of RS .



As before, the perpendicular bisector of RS contains the center of the circle. These two perpendicular bisectors meet in a unique point; that point must be the center of the circle.



17.0 Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

Problem: The vertices of a triangle PQR are the points $P(1, 2)$, $Q(4, 6)$, and $R(-4, 12)$. Which one of the following statements is true?

1. PQR is a right triangle with right $\angle P$.
2. PQR is a right triangle with right $\angle Q$.
3. PQR is a right triangle with right $\angle R$.
4. PQR is not a right triangle.

Solution: We apply the distance formula to find the lengths of the segments PQ , PR and RQ :

$$\overline{PQ} = \sqrt{(4 - 1)^2 + (6 - 2)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\overline{PR} = \sqrt{(-4 - 1)^2 + (12 - 2)^2} = \sqrt{25 + 100} = \sqrt{125} = 5\sqrt{5}$$

$$\overline{RQ} = \sqrt{(-4 - 4)^2 + (12 - 6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$$

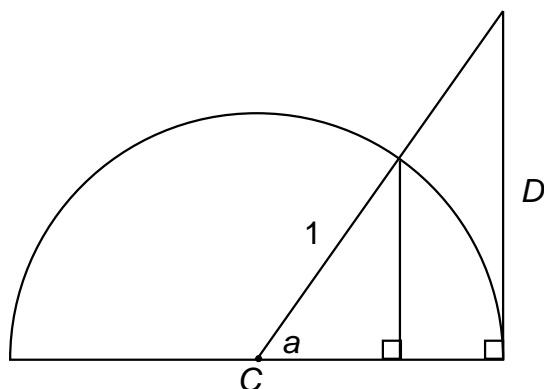
We check to see if we have a right triangle by checking the Pythagorean relation $a^2 + b^2 = c^2$, where c is the longest side of the triangle, that is $c = 5\sqrt{5}$. So we check:

$$5^2 + 10^2 = 25 + 100 = 125 = (5\sqrt{5})^2.$$

Since this is a true statement, we see that $\triangle PQR$ is indeed a right triangle with right angle $\angle Q$ (since $\angle Q$ is opposite the hypotenuse PR).

18.0 Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.

Problem: Shown below is a semicircle of radius 1 and center C . Express the unknown length D in terms of the angle a by using a trigonometric function.

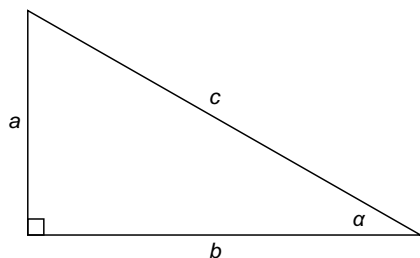


Solution: In the larger right triangle, the length of the leg adjacent to angle a must be 1 since it is also a radius of the semicircle. We can use the tangent function to obtain

$$\tan a = \frac{D}{1},$$

since the tangent of an angle in a right triangle is defined as the ratio of length of the opposite side to the length of the adjacent side. Therefore, $D = \tan a$. (*Continued on next page.*)

Further Explanation: In a right triangle as shown below, the basic trigonometric functions are defined by the ratios of certain side lengths.



In this diagram, we have

$$\begin{aligned}\sin \alpha &= \frac{\text{length of side opposite to } \alpha}{\text{length of hypotenuse}} = \frac{a}{c} \\ \cos \alpha &= \frac{\text{length of side adjacent to } \alpha}{\text{length of hypotenuse}} = \frac{b}{c} \\ \tan \alpha &= \frac{\text{length of side opposite to } \alpha}{\text{length of side adjacent to } \alpha} = \frac{a}{b}\end{aligned}$$

Note that with these definitions, the trigonometric functions are only defined for angles α with $0^\circ < \alpha < 90^\circ$.

18.0 Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.

Problem: If α is an acute angle and $\cos \alpha = \frac{1}{3}$, find $\tan \alpha$.

Solution: If α is acute, then $0^\circ < \alpha < 90^\circ$ and so $\tan \alpha$ is positive. By the trigonometric identity

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

we have

$$\sin^2 \alpha + \left(\frac{1}{3}\right)^2 = 1.$$

Therefore

$$\sin^2 \alpha = 1 - \frac{1}{9} = \frac{8}{9},$$

so that

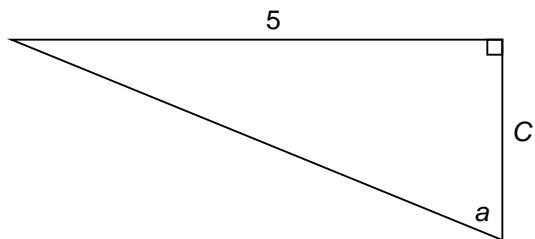
$$\sin \alpha = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

Here, we take the positive square root for $\sin \alpha$ since α is acute. Finally,

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{2\sqrt{2}/3}{1/3} = \frac{2\sqrt{2}}{3} \cdot \frac{3}{1} = 2\sqrt{2}.$$

19.0 Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.

Problem: Find the length of side C if $\angle a$ measures 70 degrees:



Solution: By the usual trigonometric ratios, we have

$$\tan a = \frac{5}{C},$$

so that

$$\tan 70^\circ = \frac{5}{C},$$

or

$$C = \frac{5}{\tan 70^\circ}.$$

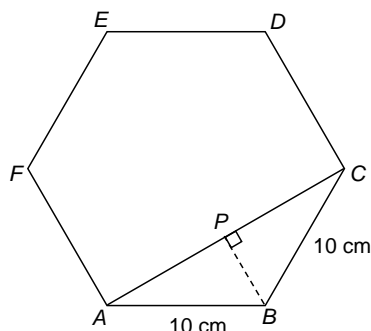
Since $\tan 70^\circ \approx 2.7475$, we have

$$C \approx \frac{5}{2.748} \approx 1.820.$$

20.0 Students know and are able to use angle and side relationships in problems with special right triangles, such as 30° , 60° , and 90° triangles and 45° , 45° , and 90° triangles.

Problem: Each side of a regular hexagon $ABCDEF$ is 10 cm long. What is the length of the diagonal AC ?

Solution: The situation is shown below.



Since $ABCDEF$ is a regular hexagon, we know that we can find the measure of $\angle ABC$ by

$$\angle ABC = \frac{(6 - 2) \cdot 180^\circ}{6} = 120^\circ.$$

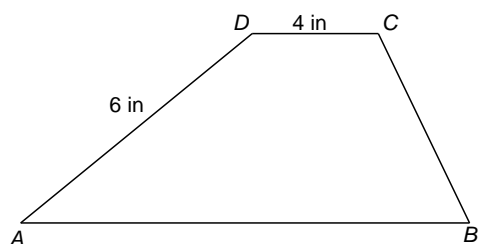
Thus, if BP is the bisector of angle $\angle ABC$, then it also bisects AC by applying SAS congruence of triangles. Therefore $\triangle ABP$ is a $30^\circ - 60^\circ - 90^\circ$ right triangle with hypotenuse of length 10 cm. In this case, the length of the longer leg \overline{AP} is equal to $\sqrt{3}/2$ times the length of the hypotenuse. That is,

$$\overline{AP} = \frac{\sqrt{3}}{2} \cdot 10 = 5\sqrt{3}.$$

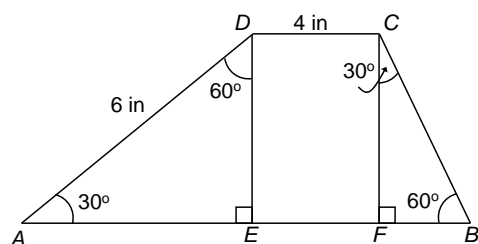
Therefore, the length of AC is twice this number, or $10\sqrt{3}$ cm.

20.0 Students know and are able to use angle and side relationships in problems with special right triangles, such as 30° , 60° , and 90° triangles and 45° , 45° , and 90° triangles.

Problem: Express the perimeter of the trapezoid $ABCD$ in the simplest exact form. Angle DAB measures 30° , and angle ABC measures 60° .



Solution: In the diagram below, the perpendiculars from C and D have been constructed to meet \overleftrightarrow{AB} at points E and F .



This makes triangles $\triangle AED$ and $\triangle CFB$ both $30^\circ - 60^\circ - 90^\circ$ right triangles. Thus, the sides are in the ratio of $1 : \sqrt{3} : 2$.

The shorter leg of the triangle AED has length one-half the length of the hypotenuse, so that $\overline{DE} = \frac{1}{2}(6 \text{ in}) = 3 \text{ in}$. The longer leg is $\sqrt{3}$ times as long as DE , so that $\overline{AE} = 3\sqrt{3} \text{ in}$.

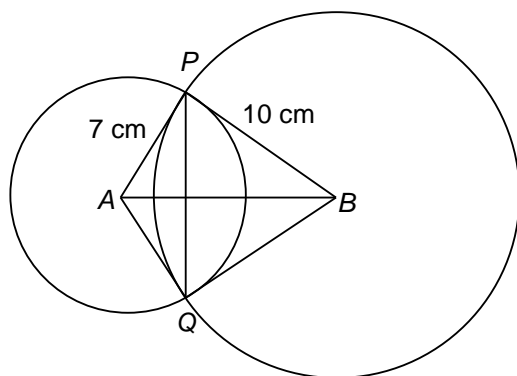
Notice that $CF \cong DE$, so that $\overline{CF} = 3 \text{ in}$ as well. Therefore, $\overline{BF} = \overline{CF}/\sqrt{3} = 3 \text{ in}/\sqrt{3} = \sqrt{3} \text{ in}$. Also, BC has length $\overline{BC} = 2 \cdot \overline{BF} = 2\sqrt{3} \text{ in}$.

Finally, $\overline{EF} = \overline{DC} = 4 \text{ in}$, since $EFCD$ is a rectangle. Since the perimeter P is the sum of the lengths of the sides, we have,

$$\begin{aligned} P &= \overline{AD} + \overline{DC} + \overline{BC} + \overline{BF} + \overline{EF} + \overline{AE} \\ &= 6 \text{ in} + 4 \text{ in} + 2\sqrt{3} \text{ in} + \sqrt{3} \text{ in} + 4 \text{ in} + 3\sqrt{3} \text{ in} \\ &= 14 + 6\sqrt{3} \text{ in.} \end{aligned}$$

21.0 Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

Problem: Two circles with centers A and B , as shown below, have radii of 7 cm and 10 cm, respectively. If the length of the common chord PQ is 8 cm, what is the length of AB ? Show all your work.



Solution: The perpendicular bisector of the chord PQ must pass through both centers of the circles, by previous results. Therefore, \overleftrightarrow{AB} must be the perpendicular bisector of segment PQ . Suppose they intersect at a point X . Since the length of chord PQ is 8 cm, we have $\overline{XP} = \frac{1}{2}(8 \text{ cm}) = 4 \text{ cm}$. Since triangle $\triangle XBP$ is a right triangle, we have

$$(\overline{XP})^2 + (\overline{XB})^2 = (\overline{BP})^2,$$

so that

$$(\overline{XB})^2 = 100 - 16 = 84,$$

which yields $\overline{XB} = 2\sqrt{21}$ cm. A similar computation with triangle $\triangle XAP$ gives

$$(\overline{XA})^2 = 7^2 - 4^2 = 33,$$

so that $\overline{XA} = \sqrt{33}$ cm. Finally,

$$\overline{AB} = \overline{XA} + \overline{XB} = \sqrt{33} + 2\sqrt{21} \approx 14.91 \text{ cm}.$$

22.0 Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

Problem: A translation maps $A(2, -3)$ onto $A'(-3, -5)$. Under the same translation, find the coordinates of B' , the image of $B(1, 4)$.

Solution: We find the translation in the x - and y -directions:

$$\Delta_x = x_{A'} - x_A = -3 - 2 = -5$$

$$\Delta_y = y_{A'} - y_A = -5 - (-3) = -2.$$

Therefore, the translation is given by $(x, y) \mapsto (x - 5, y - 2)$. To find the image B' of the point B , we simply have

$$x_{B'} = x_B - 5 = 1 - 5 = -4$$

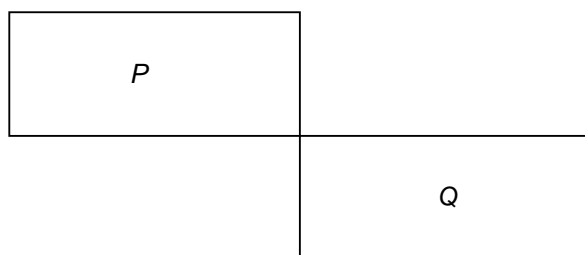
$$y_{B'} = y_B - 2 = 4 - 2 = 2.$$

Thus, the image of $B(1, 4)$ is the point $B'(-4, 2)$.

22.0 Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

Problem: Which response listed below applies to the statement that follows? The rectangle labeled Q cannot be obtained from the rectangle P by means of a:

1. Reflection (about an axis in the plane of the page)
2. Rotation (in the plane of the page)
3. Translation
4. Translation followed by a reflection



Solution:

2. If we rotate rectangle P 180° about the vertex it shares with rectangle Q , then we will obtain rectangle Q .
3. If we translate P so that its upper left vertex is located at the shared vertex of rectangles P and Q , then we will obtain rectangle Q .
4. If we translate P so that its lower left vertex is located at the shared vertex of the rectangles, and if the resulting rectangle is then reflected about the top edge of rectangle Q , then we will obtain rectangle Q .
1. Rectangle Q *cannot* be obtained from rectangle P by a reflection about an axis in the plane of the page.